NEW YORK UNIVERSITY
INSTITUTE [illegible] SCIENCES
25 Waverly Place, New York 3, N. Y.

AFOSR TN-59-945
ASTIA DOCUMENT NO.

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. BR-30

On the Dirichlet Problem For the Reduced Wave Equation

ROLF LEIS

Mathematics Division
Air Force Office of Scientific Research
Contract No AF 49(638)-229

JUNE, 1959

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. BR-30

ON THE DIRICHLET PROBLEM FOR THE REDUCED WAVE EQUATION

Rolf Leis

Rolf Leis
Rolf Leis

Sidney Borowitz
Sidney Borowitz
Acting Project Director

Qualified requestors may obtain copies of this report from the ASTIA Document Service Center, Arlington Hall Station, Arlington 12, Virginia. Department of Defense contractors must be established for ASTIA services, or have their 'need-to-know' certified by the cognizant military agency of their project or contract.

June, 1959

New York 1959

Abstract

In this paper, an existence theorem for the exterior Dirichlet problem for the reduced wave equation will be given for piecewise smooth surfaces.

Table of Contents

Page

1. Introduction 2
2. Uniqueness theorem 6
3. The setting up and discussion of the integral equations 8
4. Existence theorem 19

References 23

1. Introduction

There are three exterior boundary value problems which are well known in potential theory ($\Delta U = 0$) and in the theory of the reduced wave equation ($\Delta U + k^2 U = 0,\ k > 0$): the Dirichlet problem (the value of U is prescribed on a closed smooth surface.), the Neumann problem ($\frac{\partial U}{\partial n}$ is prescribed) and the third boundary value problem ($\frac{\partial U}{\partial n} + \delta U$ is prescribed). In all cases, it is possible to reduce the problem to integral equations and to give the solution as the sum of a single and a double layer function [1-4].

However, if the smoothness of the boundary surface is dispensed with, the integral method in its usual form is no longer applicable. In potential theory, several methods have been used to remove this difficulty[1]. In dealing with the reduced wave equation, one encounters an additional difficulty, as the radiation condition

$$\frac{\partial U}{\partial r} = ikU + o\left(\frac{1}{r}\right) \quad , \quad \text{for large } r$$

has to be satisfied.

It can be shown, however, that for a large class of piecewise smooth surfaces (regular surfaces), the integral method can be extended, using a Banach space instead of the space of continuous functions. It is the aim of this paper to give an existence theorem for the Dirichlet problem of the reduced wave equation for regular surfaces using this method.

In Section 1 , we make the necessary assumptions for the underlying surfaces and give some properties of functions to be used later. Section 2 gives a uniqueness theorem. In Section 3, we set up and discuss the integral equations. Finally, in Section 4, we give the existence theorem.

Let G_- be a bounded domain in the three dimensional Euclidian space. Let Ω be the closed surface of G_-, and G_+ be the exterior of G_-. If it is necessary to distinguish between the different sides of Ω, we do so by indices + (exterior) and - (interior). The normal vector of Ω (if it exists) is directed towards the exterior. Let Ω be orientable and regular in the following sense: (See Kellogg[5].)

Definition 1: A regular curve is continuous, has no double points, has finite length and consists of a finite number of continuously differentiable arcs.

Definition 2: A regular surface element is a set of points which, for some orientation of the axes, admits a representation

$$z = F(x,y) \quad , \quad (x,y) \in G .$$

$F(x,y)$ is twice differentiable in G. G is a closed, finite region of the (x,y)-plane, bounded by a regular curve. Furthermore, it is possible to continue the surface element beyond its boundary such that the same representation holds for a larger domain - the boundary points of the element now being interior points of that larger domain.

Definition 3: A regular surface is a set of points consisting of a finite number of regular surface elements related as follows:

1. Two regular surface elements may have in common, at most, either a vertex or an edge.
2. Three or more of the regular surface elements may have, at most, vertices in common.
3. Any two of the regular surface elements are the first and

last of a chain, such that each has an edge in common with the next.

4. All the regular surface elements having a vertex in common form a chain such that each has an edge (terminating in that vertex) in common with the next; the last may, or may not, have an edge in common with the first.

If each edge of the regular surface elements of a regular surface belongs to two of the elements, the surface is said to be closed. Here an edge of a regular surface element means one of the finite number of differentiable arcs of which its boundary is composed. A vertex is a point at which two edges meet.

Definition 4: A regular point of a regular surface is a point which, by some decomposition of the regular surface into regular surface elements, appears as an interior point of a regular surface element.

Let p be an arbitrary point in the space. Let s be a point of Ω and $r(p,s)$ be the distance between p and s. The functions

$$U(p) = \int_{\Omega} \mu(s) \frac{e^{ikr}}{r} ds \tag{1.1}$$

and

$$V(p) = \int_{\Omega} \nu(s) \frac{\partial}{\partial n_s} \frac{e^{ikr}}{r} ds \tag{1.2}$$

are solutions of the reduced wave equation for all $p \in G_+$. We use the following results obtained in potential theory [6].

Lemma 1: Let $\mu(s)$ be continuous. Then $U(p)$ is continuous everywhere. Let s be a regular point of Ω. Then we get the normal derivatives of U in s by the 'jump relations'

$$\frac{\partial U}{\partial n}(s)_+ = -2\pi\mu(s) + \int_\Omega \mu(s') \frac{\partial}{\partial n_s} \frac{e^{ikr}}{r(s,s')} \, ds'$$

$$\frac{\partial U}{\partial n}(s)_- = 2\pi\mu(s) + \int_\Omega \mu(s') \frac{\partial}{\partial n_s} \frac{e^{ikr}}{r(s,s')} \, ds' \quad .$$

Lemma 2: a) Let $\nu(s)$ be continuous. Then we get, in regular points s,

$$V(s)_+ = 2\pi \, \nu(s) + \int_\Omega \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} \, ds'$$

$$V(s)_- = -2\pi \, \nu(s) + \int_\Omega \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} \, ds' \quad .$$

b) Let $\nu(s)$ be continuous in regular points; then

$$\int_\Omega \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} \, ds' = \frac{1}{2}(V_+ + V_-) = W(s)$$

is Hölder continuous in regular points s.

c) Let $\nu(s)$ be Hölder continuous in regular points Ω. Then the derivatives of $W(s)$ exist and are Hölder continuous in regular points of Ω.

d) Let $\nu(s)$ be Hölder continuously differentiable in regular points of Ω. Then the normal derivative of $V(p)$ exists in regular points of Ω and is Hölder continuous.

Lemma 3: Let $\nu(s)$ be continuous in regular points. Furthermore, let $\nu(s)$ be a solution of the integral equation

$$\nu(s) = \lambda \int_\Omega \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} ds' .$$

Then the normal derivative of $V(p)$ exists in regular points of Ω and is Hölder continuous.

Next we prove

Lemma 4: The function

$$A(s) = \int_\Omega \left| \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} \right| ds'$$

is bounded.

We shall prove boundedness in non-regular points. Boundedness in regular points will follow in the same manner from Lemma 2b. Thus let us assume that Σ_1 and Σ_2 are two adjacent surface elements, with a near edge or vertex. Then we have

$$(1.3) \qquad A(s) \leq C + \int_{\Sigma_1} \left| \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r} \right| ds' + \int_{\Sigma_2} \left| \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r} \right| ds' .$$

According to our assumption, both Σ_1 and Σ_2 can be continued to $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ such that boundary points of Σ_j become interior points of $\overline{\Sigma}_j$. Thus, we get

$$(1.4) \qquad |A(s)| \leq C + \int_{\overline{\Sigma}_1} \left|\frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r}\right| ds' + \int_{\overline{\Sigma}_2} \left| \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r} \right| ds' \leq C_1$$

using the jump relations (Lemma 2a). This proves Lemma 4.

Throughout this paper, we shall interpret the surface integrals in Lebesgue's sense. Therefore, we shall consider functions to be equal if they differ only on a set of measure zero. As the edges and vertices of our regular surface Ω are finite, their measure is zero. Thus, we shall say that two functions, defined on Ω, are equal if they coincide in all regular points of Ω.

2. Uniqueness theorem

In this section, we shall prove

Theorem 1: There is, at most, one function $U(p)$ which satisfies the following conditions:

1. $U(p)$ is twice differentiable in G_+ ;
2. In G_+ $U(p)$ satisfies the differential equation

$$\Delta U + k^2 U = 0 \; ; \; k > 0 \quad ;$$

3. $U(p)$ satisfies the radiation condition uniformly for all directions

$$\frac{\partial U}{\partial r} = ikU + o\left(\frac{1}{r}\right) \quad \text{for large } r.$$

4. Let $g(s)$ be a continuous function on Ω. Then $U(p)$ satisfies the boundary condition $U(s) = g(s)$, provided that for $\Omega_n \in G_+$ and $\Omega_n \to \Omega$,

$$\int_{\Omega_n} U \frac{\partial \overline{U}}{\partial n} \, ds \to \int_{\Omega} U \frac{\partial \overline{U}}{\partial n} \, ds .$$

Let us assume that we have two different functions satisfying the assumptions of Theorem 1. We take their difference $W(p)$ which vanishes in regular points of Ω. Let us take a sufficiently large sphere Σ with radius R containing Ω and let G be the region bounded by Ω and Σ. From the radiation condition it follows

$$(2.1) \qquad \left(\frac{\partial W}{\partial n} - ikW\right)\left(\frac{\partial \overline{W}}{\partial n} + ik\overline{W}\right) = o\left(\frac{1}{R^2}\right)$$

or

$$(2.2) \qquad \int_{\Sigma} \left\{ \left|\frac{\partial W}{\partial n}\right|^2 + k^2|U|^2 - ik\left(W \frac{\partial \overline{W}}{\partial n} - \overline{W} \frac{\partial W}{\partial n}\right) \right\} ds = o(1) .$$

Let Ω_n now be a sequence of regular surfaces in G approaching Ω. Then, by Green's formula, we get

$$(2.3) \qquad \int_{\Omega_n} \left(W \frac{\partial \overline{W}}{\partial n} - \overline{W} \frac{\partial W}{\partial n}\right) ds = \int_{\Sigma} \left(W \frac{\partial \overline{W}}{\partial n} - \overline{W} \frac{\partial W}{\partial n}\right) ds .$$

According to our assumptions, the limit $\Omega_n \to \Omega$ in (2.3) exists and is equal to zero. Thus we get

$$(2.4) \qquad \int_{\Sigma} \left\{ \left|\frac{\partial W}{\partial n}\right|^2 + k^2|U|^2 \right\} ds = o(1) .$$

According to a result obtained by Rellich[7], from

$$(2.5) \qquad \int_{\Sigma} |U|^2 \, ds = o(1) \qquad \text{for } R \to \infty$$

it follows that $U \equiv 0$ in G_+ .

Thus, from (2.4), $W(p)$ vanishes identically in G_+ . This proves Theorem 1.

3. The setting up and discussion of the integral equations

We define the functions

$$(3.1) \qquad Q(p,s,) = \frac{1}{2\pi} \frac{e^{ikr}}{r(p,s)} \quad , \quad K(p,s,) = \frac{\partial}{\partial n_s} Q(p,s)$$

which solve for $p \neq s$ the differential equation

$$(3.2) \qquad \Delta U + k^2 U = 0 .$$

The functions Q and K are fundamental solutions of our problem. We shall try to construct solutions of our boundary value problem with the help of these functions by integration over Ω. Thus we consider the functions ($\nu(s)$ and $\mu(s)$ being continuous)

$$U_1(p) = \int_\Omega \mu(s)\, Q(p,s)\, ds \tag{3.3}$$

and

$$U_2(p) = \int_\Omega \nu(s)\, K(p,s)\, ds . \tag{3.4}$$

We remark that the function $K(p,s)$ is not symmetric. As $K(p,s)$ or $K(s',s)$ always occur in integrals, we sometimes (to prevent misunderstanding) add indices 1 or 2 to K (e.g., $K_1(p,s)$) denoting that the first (or second) variable is free.

To solve the Dirichlet problem, we assume that $U = U_2$ and get, according to Lemma 2, in regular points,

$$g(s) = \nu(s) + \int_\Omega \nu(s')\, K(s,s')\, ds' . \tag{3.5}$$

For (3.5) we write, symbolically,

$$g = \nu + K_1 \nu \tag{3.6}$$

To solve the Neumann problem, we make the assumption that $U = U_2$ and, according to Lemma 1, get in regular points

$$g(s) = - \mu(s) + \int_\Omega \mu(s')\, K(s',s)\, ds' . \tag{3.7}$$

For (3.7) we write, symbolically,

$$g = -\mu + K_2 \mu . \tag{3.8}$$

Thus, in regular points adjoint integral equations were obtained for the unknown layer functions μ and ν. We shall now take these integral equations as a basis for all points of Ω. Of course, this is not possible in the space of continuous functions, as $K_2\nu$ becomes logarithmically singular at the edges [8]. So we have to use suitable Banach spaces for our discussion.

The following results hold for integral equations in Banach spaces [9], [10]:

1. Let A be a transformation of the Banach space $\mathcal{B}$. If A is linear and bounded, so is the adjoint transformation A^* of the adjoint space $\mathcal{B}^*$.
2. If A is completely continuous, so is A^*. (An operator is called completely continuous if it maps a bounded into a compact sequence.)
3. Let A be completely continuous: $f, g \in \mathcal{B}$ and $F, G \in \mathcal{B}^*$. Then the Fredholm alternative is valid for the integral equations $g = f + Af$ and $G = F + A^*F$. In particular, the number of linearly independent solutions of the homogeneous equations is the same and is finite.

In the space L^p of functions the p-th power of which is absolutely integrable, the following criterion for compactness holds [11].

Lemma 5: $M \in L^p$ is compact provided that for all $x \in M$

1. $\|x\| \leq k$
2. for each $\epsilon > 0$ exists a $\delta > 0$ such that for all h with $|h| \leq \delta$

$\|x(t + h) - x(t)\| \leq \epsilon$ holds.

Applying these results, we shall show that the Fredholm alternative holds for the equations

$$g = f + K_1 f \quad \text{and} \quad G = F + K_2 F . \tag{3.9}$$

To do so we choose the Banach space L^1 for the second equation $(F,G \in L^1)$. It is known that $L^{1*} = L^\infty$ (the space of integrable and almost everywhere bounded functions). Thus $f,g \in L^\infty$.

First we have to show that K_2 is a bounded operator in L^1. To do this we form

$$(3.10) \qquad \|K_2F\| = \int_\Omega \left| \int_\Omega F(s')K(s',s)ds' \right| ds \leq \iint_\Omega |F(s')| \; |K(s',s)| ds'ds$$

$$= \iint_\Omega |F(s')| \; |K(s',s)| dsds' = \int_\Omega |F(s')| \; A(s')ds' \leq c \|F\|$$

according to Lemma 4. It is possible to change the order of integration. According to a theorem by Lusin[12], there exists for every number $\epsilon > 0$ a surface $\Omega_\epsilon \subset \Omega$ with $|\Omega - \Omega_\epsilon| \leqq \epsilon$ such that $F(s)$ is continuous in Ω_ϵ. Thus changing is possible in Ω_ϵ (assuming that Ω_ϵ consists of regular points only). However, the estimate obtained in (3.10) does not depend on ϵ, so that we can take the limit $\epsilon \to 0$ (as is customary, introducing improper integrals).

Next we prove that K_2 is completely continuous. Let $M \subset L^1$ be a bounded set of functions $F \in M$, $\|F\| \leqq C$. We have to show that K_2F is compact. To do so we use Lemma 5. It follows from (3.10) that

$$(3.11) \qquad \|K_2F\| \leqq c\,C \leqq k \qquad \text{for all } F \in M.$$

Thus we form

$$(3.12)\qquad \|(K_2F)(t+h)-(K_2F)(t)\| = \int_\Omega \left| \int_\Omega F(s') \left\{K_2(s',t+h)-K_2(s',t)\right\} ds' \right| dt$$

$$\leq \int_\Omega |F(s')| B\,(s',h)\, ds'$$

with

$$(3.13)\qquad B(s',h) = \int_\Omega |K_2(s',t+h) - K_2(s',t)|\, dt\ .$$

In order to discuss $B(s',h)$, we choose all points of Ω having, from edges or vertices, a distance $\leq \zeta = 3\sqrt[4]{|h|}$ (h sufficiently small) and denote them by Z_ζ. Thus we can split up Ω in the form $\Omega = \Omega^* + Z_\zeta$.

For all points $s' \in Z_\zeta$ there holds, according to Lemma 4:

$$(3.14)\qquad |B\,(s',h)| \leq \alpha < \infty \quad \text{for all h.}$$

So let us assume $s' \in \Omega^*$. We choose a sphere around s' with radius $\tau (2|h|<2\tau<\zeta)$. We denote the part of Ω^* lying in this sphere by S_τ. Then we get

$$(3.15)\quad \int_{S_\tau} |K_2(s',t+h)-K_2(s',t)|dt \leq \int_{S_\tau} |K_2(s',t+h)|dt+\int_{S_\tau} |K_2(s',t)|dt = \mathcal{O}(\tau)\ .$$

According to the mean value theorem of differential calculus, we get

$$(3.16)\quad \int_{\Omega-S_\tau} |K_2(s',t+h)-K_2(s',t)|dt \leq |h| \max \int_{\Omega-S_\tau} \left|\frac{\partial}{\partial t} K_2(s',t+\vartheta h)\right|dt \leq |h|\, \mathcal{O}\!\left(\frac{1}{(\tau-|h|)^3}\right)\ .$$

Thus we have for all $s' \in \Omega^*$

(3.17) $$|B(s',h)| \leq \mathcal{O}\left(\tau + \frac{|h|}{(\tau-|h|)^3}\right)$$

or if we choose $|h| = \tau^4$ for all $|h| < (\frac{1}{2})^{\frac{4}{3}}$

(3.18) $$|B(s',h)| \leq \beta|h|^{\frac{1}{4}} \qquad \text{for all } s' \in \Omega^*.$$

Continuing (3.12) we arrive at

(3.19) $$\|(K_2F)(t+h)-(K_2F)(t)\| \leq \alpha \int_{Z_\zeta} |F(s')|ds' + \beta|h|^{\frac{1}{4}} \|F\| .$$

As the indefinite integral

(3.20) $$\int |F(s')|ds'$$

is a totally continuous function, we can find for every given $\epsilon > 0$ a $\delta_1 > 0$ such that for all $|h| = (\frac{1}{3}\zeta)^4 \leq \delta_1$

(3.21) $$\int_{Z_\zeta} |F(s')|ds' \leq \frac{\epsilon}{2\alpha}$$

holds. Furthermore, for all $|h| \leq \delta_2$ we have

(3.22) $$\beta \; C \; h^{\frac{1}{4}} \leq \frac{\epsilon}{2} .$$

Thus, if we choose $|h| \leqq \delta = \text{Min}\,(\delta_1,\delta_2)$ we finally get

$$\|(K_2F)(t+h)-(K_2F)(t)\| \leqq \epsilon \qquad \text{for all } F \in M \tag{3.23}$$

which proves that K_2 is completely continuous.

Finally we want to show that $K_2^* = K_1$. Every functional on L^1 can be written as

$$(F,f) = \int_\Omega F(s)\; f(s)\; ds \tag{3.24}$$

with $f \in L^\infty$. Thus

$$\begin{aligned}(K_2F,f) &= \int_\Omega f(s) \int_\Omega K(s',s)\; F(s')\; ds'ds \\ &= \int_\Omega F(s') \int_\Omega f(s)\; K(s',s)\; ds\; ds' = (F,K_2^{\,*}f)\end{aligned} \tag{3.25}$$

or

$$K_2^* = K_1 \,. \tag{3.26}$$

Thus the Fredholm alternative holds for the integral equations

$$g = f + K_1f \quad \text{and } G = F + K_2F \qquad g,f \in L^\infty\,; \quad G,F \in L^1\,. \tag{3.27}$$

Before giving an explicit solution of the Dirichlet problem, let us discuss the solutions of the homogeneous equations. First we take

$$(3.28) \qquad 0 = f + K_1 f.$$

We want to show that f is continuous in regular points. Let s_1 be a regular point of Ω and $|s - s_1| = h$. Let the shortest distance of s_1 vertex be ρ. Then we choose a sphere around s_1 with radius $\tau (2h < \tau < \frac{\rho}{2})$. Let $F_\tau(s_1)$ be the part of Ω within this sphere. Then we have,

$$(3.29) \qquad \int_{F_\tau(s_1)} |f(s')| \; |K_1(s_1,s')-K_1(s,s')| ds' \leq \int_{F_\tau(s_1)} |f(s'| \; |K_1(s_1,s')| \; ds'$$

$$+ \int_{F_\tau(s_1)} |f(s')| \; |K_1(s,s')| \; ds' = \mathcal{O}(\tau)$$

and

$$(3.30) \qquad \int_{\Omega - F_\tau(s_1)} |f(s')| \; |K_1(s_1,s')-K_1(s,s')| \; ds' = \mathcal{O}\left(\frac{h}{\tau^3}\right).$$

If we choose $\tau^4 = h$, we get for all $h < \mathrm{Min} \left((\frac{1}{2})^{\frac{4}{3}}, \frac{\rho}{2} \right)$

$$(3.31) \qquad |(K_1 f)(s_1) - (K_1 f)(s)| \leq C \; |s-s_1|^{\frac{1}{4}}$$

which proves Hölder continuity. At edges or vertices, of course, f is no longer continuous (jump relation). Thus, according to Lemma 2, we get

Lemma 6: In regular points, the solutions of the integral equation $f = K_1 f$ are Hölder continuously differentiable.

Next we prove

Lemma 7: The solutions of the integral equation $F = K_2 F$ are Hölder continuously differentiable in regular points.

To prove Lemma 7, we have to show that the solutions F are bounded in regular points. When this is established, we argue as we did in (3.29f). To prove boundedness, we use the following argument. Let s be a regular surface element of Ω. The integrals

$$(3.32) \qquad (g,F)_s = \int_s g(s)\, F(s)\, ds$$

may exist for all $g \in L_s^p$ (p fixed $1 \leqq p \leqq \infty$). Then $F \in L_s^q$ $(\frac{1}{p} + \frac{1}{q} = 1)$. This is certainly true for $p = \infty$ and $q = 1$, as $F \in L^1$. Thus let us assume that $p < \infty$. Then $(g,F)_s$ defines a functional over L_s^p which can be written as

$$(3.33) \qquad (g,F)_s = (g,\phi)_s \qquad \phi \in L_s^q$$

or with $\phi_1 = F - \phi$, $\phi_1 \in L_s^1$

$$(3.24) \qquad (g,\phi_1)_s = 0 \qquad \text{for all } g \in L_s^p.$$

Now we choose

$$(3.35) \qquad g_1(s) = \begin{cases} 1 & \text{for s with } \phi_1(s) = 0 \\ \\ \text{sign } \phi_1 & \text{for s with } \phi_1(s) \neq 0 \end{cases} .$$

Then we have $|g_1(s)| = 1$ or $g_1 \in L^p_s$ and

$$(3.36) \qquad (g_1,\phi_1)_s = \int_S |\phi_1| \, ds = 0 ,$$

and it follows that ϕ_1 vanishes or $F \in L^q_s$.

We now prove Lemma 7 by iterating this argument. Let s_1 be a regular point and Σ be a neighborhood (e.g., a sphere with radius R) of s_1 lying entirely in a regular surface element of Ω . We choose

$$(3.37) \qquad g \in L^p_\Sigma \;\; ; \;\; p = 2 + \epsilon, \; \epsilon > 0 \;\; , \;\; g = 0 \text{ elsewhere.}$$

Then we have

$$(3.38) \quad (g,F)_\Sigma = \iint_\Omega g(s) \; F(s') \; K(s',s) \; ds'ds = \int_\Omega F(s') \left\{ \int_\Omega g(s) \; K(s',s) \; ds \right\} ds'$$

and with $q = 1 + \alpha \;\; (\alpha < 1)$

$$(3.39) \qquad \left| \int_{\Omega \text{ or } \Sigma} g(s)K(s',s)ds \right| \leqq \left| \int_\Sigma |g|^p ds \right|^{\frac{1}{p}} \left| \int_\Sigma |K(s',s)|^q ds \right|^{\frac{1}{q}} < \infty \quad .$$

Thus we obtain the result that $F \in L^q_\Sigma$.

Next we choose $\Sigma_1 \subset \Sigma$ (e.g., a sphere with radius $\frac{R}{2}$) and $g \in L_{\Sigma_1}^{\frac{p+1}{2}}$, $g = 0$ elsewhere. Then

$$(3.40) \qquad (g,F)_{\Sigma_1} = \iint_\Omega g(s) \; F(s') \; K(s',s) \; ds' \; ds$$

$$= \int_\Omega F(s') \left\{ \int_\Omega g(s) \; K(s',s) \; ds \right\} ds' \quad = \quad (F,G)_\Omega \quad .$$

Now we have

$$(3.41)\qquad |G(s')|^p \leqq \left\{ \int_{\Sigma_1} \sqrt{|g|}\, (\sqrt{|g|}\ |K|)^{\frac{1}{p}} (\sqrt{|g|}\ |K|)^{\frac{1}{q}} \, ds \right\}^p$$

$$\leqq \int_{\Sigma_1} \sqrt{|g|}^{\,p} (\sqrt{|g|}\ |K|)\, ds \left\{ \int_{\Sigma_1} \sqrt{|g|}\ |K|\ ds \right\}^{\frac{p}{q}}$$

$$\leqq C \int_{\Sigma_1} \sqrt{|g(s)|}^{\,p+1} |K(s',s)|\ ds = C \int_{\Sigma} \sqrt{|g|}^{\,p+1} |K|\ ds$$

or

$$(3.42)\qquad G \in L^p_\Sigma$$

On the other hand, G is bounded in $\Omega - \Sigma$, because g is only in $\Sigma_1 \subset \Sigma$ different from zero. Thus

$$(3.43)\qquad (F,G)_\Omega = (F,G)_\Sigma + (F,G)_{\Omega-\Sigma} < \infty$$

because $F \in L^q_\Sigma$ and $F \in L^1_{\Omega-\Sigma}$. The corresponding q to $\frac{p+1}{2} = \frac{3+\epsilon}{2}$ is $1 + \frac{2}{1+\epsilon}$. Therefore $(\epsilon = \frac{1}{2})$

$$(3.44)\qquad F \in L^{\frac{7}{3}}_{\Sigma_1} ,$$

and it follows with $p = \frac{7}{4}$; $q = \frac{7}{3}$

$$(3.45)\qquad \left| \int_{\Sigma_1} F(s')\, K(s',s)\, ds' \right| \leqq \left| \int_{\Sigma_1} |F|^{\frac{7}{3}} \right|^{\frac{3}{7}} \left| \int_{\Sigma_1} |K|^{\frac{7}{4}} \right|^{\frac{4}{7}} < \infty .$$

This means that $F(s)$ is bounded in a neighborhood of $s = s_1$ which proves Lemma 7.

4. Existence theorem

In this section, we shall prove

Theorem 2: There exists a function, $U(p)$, which satisfies the following conditions:

1. $U(p)$ is twice differentiable in G_+ ;
2. In G_+ , $U(p)$ satisfies the differential equation

$$U + k^2 U = 0 \quad k > 0 ;$$

3. $U(p)$ satisfies the radiation condition uniformly for all directions

$$\frac{\partial U}{\partial r} = i\,k\,U + o\left(\frac{1}{r}\right) \qquad \text{for large } r ;$$

4. Let $g(s)$ be a continuous function on Ω . Then $U(p)$ satisfies the boundary condition $U(s) = g(s)$.

From now on the proof of the existence theorem proceeds in a way similar to that of the smooth case[3]. Thus we can deal with it rather briefly. In the case of the Dirichlet problem, we have the integral equation

$$g = f + K_1 f. \tag{4.1}$$

Let $\Phi = \{\phi\}$ be the set of homogeneous solutions

$$0 = \phi + K_1\phi \qquad \phi \in L^\infty \tag{4.2}$$

and $H = \{\eta\}$ be the set of homogeneous solutions of the adjoint equation

$$0 = \eta + K_2\eta \qquad \eta \in L^1. \tag{4.3}$$

Consider the function

$$V(p) = \int_\Omega \eta Q(p,s)\, ds . \tag{4.4}$$

The limit

$$(4.5)\qquad \int_{\Omega_n} V(s)\,ds \to \int_{\Omega} V(s)\,ds$$

exists for a sequence $\Omega_n \to \Omega$ (Take, e.g., parallel surfaces Ω_p and consider every regular surface element Σ, $\Sigma_p \to \Sigma$ alone.) The same is true for $\frac{\partial V}{\partial n} \left(\frac{\partial V}{\partial n}(s) \in L^1 \right)$ because

$$(4.6)\qquad \frac{1}{2\pi} \int_{\Sigma_p} \frac{\partial}{\partial n} \int_{\Sigma} \eta \frac{1}{r}\,ds\,ds' = \int_{\Sigma} \eta g(s,p)\,ds$$

with

$$(4.7)\qquad g(s,p) = \frac{1}{2\pi} \int_{\Sigma_p} \frac{\partial}{\partial n_{s'}} \frac{1}{r}\,ds'$$

which is continuous for all s and p. Thus

$$(4.8)\qquad \int_{\Sigma_p} \frac{\partial V}{\partial n}\,ds \to \int_{\Sigma} \frac{\partial V}{\partial n}\,ds\ .$$

We can now apply Green's formula on G_- bounded by Ω with the functions $V(q)$ and $Q(p,q)$ $p \in G_+$ and get

$$(4.9)\qquad 0 = \int_{\Omega_-} \left\{ V(s)\,K(p,s) - \frac{\partial V}{\partial n}(s)\,Q(p,s) \right\} ds\ .$$

In regular points of Ω, we have

$$(4.10)\qquad \left.\frac{\partial V}{\partial n}\right|_- = \eta + K_2\eta = 0.$$

Thus from (4.9) in regular points, we get

$$(4.11) \qquad 0 = V + K_1 V$$

which means that $V \in \Phi$. The same argument gives $\overline{V} \in \Phi$. Thus (analogous to (4.6))

$$(4.12) \qquad \int_{\Sigma_p} V \frac{\partial \overline{V}}{\partial n} \, ds \to \int_{\Sigma} V \frac{\partial \overline{V}}{\partial n} \, ds ,$$

and the uniqueness theorem holds for $V(p)$. We define a mapping $M\eta = \phi = V(s)$. This mapping is one to one. For suppose $\phi = 0$, then $V \equiv 0$ in G_+. This means

$$(4.13) \qquad \left.\frac{\partial V}{\partial n}\right|_+ = - \eta + K_\eta = 0$$

or, together with (4.10), $\eta = 0$. Applying Green's formula on $V(p)$ and $\overline{V}(p)$ in G_+ we get, using the radiation condition,

$$(4.14) \qquad \operatorname{Im} \int_{\Omega_+} V \frac{\partial \overline{V}}{\partial n} \, ds \neq 0 \qquad \text{except } V \equiv 0$$

or

$$(4.15) \qquad \operatorname{Im} \int_{\Omega} \eta \, \overline{M\eta} \, ds \neq 0 \qquad \text{except } \eta = 0.$$

The matrix (ϕ^i being a basis of Φ)

$$(4.16) \qquad C^{ik} = \int_{\Omega} \phi^i \eta^k \, ds = (\phi^i, \eta^k)$$

has a non zero determinant. Otherwise a $\tilde{\eta} \in H$ would exist with

$$(4.17) \qquad (\phi_i, \tilde{\eta}) = 0 \qquad \text{for all } \phi_i .$$

Or, if we choose $\phi_i = \overline{M\tilde{\eta}}$

$$(4.18) \qquad (\overline{M\tilde{\eta}} , \eta) = 0$$

which means $\eta = 0$. Thus we can find a $\phi^* \in \Phi$ such that

$$(4.19) \qquad (\eta, g - \phi^*) = 0 \qquad \text{for all } \eta \in H.$$

The integral equation

$$(4.20) \qquad g - \phi^* = f + K_1 f$$

can now be solved. Let η^* be defined by $\phi^* = M\eta^*$. The function

$$(4.21) \qquad U(p) = \int_\Omega f(s)\, K(p,s)\, ds + \int_\Omega \eta^*(s)\, Q(p,s)\, ds$$

then solves the Dirichlet problem.

References

[1] Kellog, O.D. - Foundations of Potential Theory; New York, Dover, 1952.

[2] Weyl, H. - Math. Zeits.; 55, 187 (1952).

[3] Müller, C. - Math. Zeits.; 56, 80(1952).

[4] Leis, R. - A.R.M.A.; 2 , 101 (1958).

[5] Kellogg, O.D. - l.c., p. 29 ff.

[6] Lichtenstein, L. - Enz. Math. Wiss.; II/3, (1924).

[7] Rellich, F. - Jahresberichte d. dt. Math. Ver.; 53, 57 (1943).

[8] Leis, R. - N.Y.U., Inst.Math.Sci.,Div.EM Res., Research Report No. BR-29, 1959

[9] Ljusternik, L.A. and W.I. Sobolew - Elemente de Funktionalanalysin; Berlin, Akademie Verlag, 1955.

[10] Riesz, F. and S.Z. Nagy - Functional Analysis; New York, Frederick Ungar Publ. Co., 1955, p. 16.

[11] Ljusternik, L.A. and W.I. Sobolew - l.c., p. 45

[12] Kamke, E. - Das Lebesque-Stieltjes Integral; Leipzig, Teubner, 1956.

Distribution List for Research Reports

Contract No. AF 49(638)229

(ONE copy unless otherwise noted)

(3)Commander
European Office, ARDC
47 Rue Cantersteen
Brussels, Belgium

Applied Mathematics and Statistics Lab.
Stanford University
Stanford, California

Department of Mathematics
University of California
Berkeley, California

Commander
Air Force Flight Test Center
ATTN: Technical Library
Edwards Air Force Base, California

The Rand Corporation
Technical Library
1700 Main Street
Santa Monica, California

Director of Advanced Studies
Air Force Office of Scientific Research
Post Office Box 2035
Pasadena 2, California

Commander
Western Development Division
ATTN: WDSIT
P.O. Box 262
Inglewood, California

Department of Mathematics
Yale University
New Haven, Connecticut

Chief of Naval Operations
Department of the Navy
ATTN: OP03EG; Mr. Arthur Kaufman
Washington 25, D.C.

(2)Office of Naval Research
Department of the Navy
ATTN: Code 432
Washington 25, D.C.

Department of Commerce
Office of Technical Services
Washington 25, D.C.

Director National Security Agency
ATTN: Dr. H.H. Campaigne
Washington 25, D.C.

Library
National Bureau of Standards
Washington 25, D.C.

National Applied Mathematics Labs.
National Bureau of Standards
Washington 25, D.C.

Headquarters, USAF
Assistant for Operations Analysis
Deputy Chief of Staff, Operations, AFOOA
Washington 25, D.C.

(2)Commander
Air Force Office of Scientific Research
ATTN: SRDB
Washington 25, D.C.

(2)Commander
Air Force Office of Scientific Research
ATTN: SRE
Washington 25,D.C.

National Science Foundation
ATTN: Dr. Leo Cohen
Washington 25, D.C.

Commander
Air Force Armament Center
ATTN: Technical Library
Eglin Air Force Base, Florida

Commander
Air Force Missile Test Center
ATTN: Technical Library
Patrick Air Force Base, Florida

Department of Mathematics
Northwestern University
Evanston, Illinois

Institute for Air Weapons Research
Museum of Science and Industry
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Illinois
Urbana, Illinois

Department of Mathematics
Purdue University
Lafayette, Indiana

Institute for Fluid Dynamics
and Applied Mathematics
University of Maryland
College Park, Maryland

Mathematics and Physics Library
The Johns Hopkins University
Baltimore, Maryland

Commander
HQ Air Research and Development Command
Director of Research
Andrews Air Force Base
Washington 25, D.C.

Department of Mathematics
Harvard University
Cambridge 38, Massachusetts

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Geophysics Research Library
L.G. Hanscom Field
Bedford, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Electronics Research Library
L. G. Hanscom Field
Bedford, Massachusetts

Psychological Labs.
ATTN: Dr. George A. Miller
Memorial Hall
Harvard University
Cambridge 38, Mass.

Department of Mathematics
Wayne University
ATTN: Dr. Y.W. Chen
Detroit 1, Michigan

Willow Run Research Center
University of Michigan
Ypsilanti, Michigan

Department of Mathematics
Folwell Hall
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Institute of Technology
Engineering Building
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Washington University
St. Louis 5, Missouri

Department of Mathematics
University of Missouri
Columbia, Missouri

Linda Hall Library
ATTN: Mr. Thomas Gillis
Document Division
5109 Cherry Street
Kansas City 10, Missouri

Commander
Strategic Air Command
ATTN: Operations Analysis
Offutt Air Force Base
Omaha, Nebraska

The James Forrestal Res. Center Lib.
Princeton University
Princeton, New Jersey

Library
Institute for Advanced Study
Princeton, New Jersey

Department of Mathematics
Fine Hall
Princeton University
Princeton, New Jersey

Commander
AF Missile Development Center
ATTN: Technical Library
Holloman Air Force Base, New Mexico

Commander
Air Force Special Weapons Center
ATTN: Technical Library
Kirtland Air Force Base
Albuquerque, New Mexico

Prof. J. Wolfowitz
Mathematics Department
White Hall
Cornell University
Ithaca, New York

Department of Mathematics
Syracuse University
Syracuse, New York

New York University
Institute of Mathematical Sciences
Division of Electromagnetic Research
ATTN: Professor M. Kline
25 Waverly Place
New York 3, N.Y.

Department of Mathematics
Columbia University
ATTN: Professor B.O. Koopman
New York 27, N.Y.

Department of Mathematical Statistics
Fayerweather Hall
ATTN: Dr. Herbert Robbins
Columbia University
New York 27, N.Y.

Commander
Rome Air Development Center
ATTN: Technical Library
Griffiss Air Force Base
Rome, New York

Institute for Aeronautical Sciences
ATTN: Librarian
2 East 64th Street
New York 21, N.Y.

Institute of Statistics
North Carolina State College of A and E
Raleigh, North Carolina

Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina

(2)Office of Ordnance Research
Box CM
Duke Station
Durham, North Carolina

Department of Mathematics
Duke University
Duke Station
Durham, North Carolina

Commander
Air Technical Intelligence Center
ATTN: ATIAE-4
Wright-Patterson Air Force Base, Ohio

Commander
Wright Air Development Center
ATTN: Technical Library
Wright-Patterson Air Force Base, Ohio

(2)Commander
Wright Air Development Center
ATTN: ARL Technical Library, WCRR
Wright-Patterson Air Force Base, Ohio

(2)Commandant
USAF Institute of Technology
ATTN: Technical Library, MCLI
Wright-Patterson Air Force Base, Ohio

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania

Commander
Arnold Engineering Development Center
ATTN: Technical Library
Tullahoma, Tennessee

Defense Research Laboratory
University of Texas
Austin, Texas

Department of Mathematics
Rice Institute
Houston, Texas

Commander
AF Personnel and Training Research Center
ATTN: Technical Library
Lackland Air Force Base
San Antonio, Texas

(10)Armed Services Technical Information Agency
Arlington Hall Station
Arlington 12, Virginia

Department of Mathematics
University of Wisconsin
Madison, Wisconsin

Mathematics Research Center
ATTN: R.E. Langer
University of Wisconsin
Madison, Wisconsin

Naval Electronics Laboratory
San Diego 52, California

Research Laboratory of Electronics
Mass. Institute of Technology
Rm. 20B-221
Cambridge 39, Mass.

Dr. H.G. Booker
School of Electrical Engineering
Cornell University
Ithaca, N.Y.

Department of Mathematics
Carnegie Institute of Technology
Pittsburgh, Penna.

Signal Corps Engineering Labs.
Technical Documents Center
Evans Signal Laboratory
Belmar, N.J.

Commanding General
Signal Corps Engineering Labs.
ATTN: Technical Reports Library
Fort Monmouth, N.J.

Chief of Staff
HQ United States Air Force
Pentagon, Washington 25, D.C.
ATTN: AFMRD - 5

Office of Chief Signal Officer
Signal Plans and Operations Div.
Com. Liaison Br., Radio Prop. Section
The Pentagon, Washington 25, D.C.
ATTN: SIGOL - 2, Rm.2C

Library
Boulder Laboratory
National Bureau of Standards
Boulder, Colorado

Naval Research Laboratory
Washington 25, D.C.
ATTN: Technical Data Section

Air Force Development Field Rep.
Naval Research Laboratory
Code 1110
Washington 25, D.C.

Ballistics Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
ATTN: Dr. Keats Pullen

Mass. Institute of Technology
Lincoln Laboratory
P.O. Box 390
Cambridge 39, Mass.
ATTN: Dr. T.J. Carroll

Mr. Keeve Siegel
Willow Run Research Center
Willow Run Airport
Ypsilanti, Michigan

Willow Run Research Center
University of Michigan
Willow Run Airport
ATTN: Dr. C.L. Dolph

Director
Naval Research Laboratory
Code 3480
Washington 25, D.C.
ATTN: Dr. L.C. van Atta

Director
Naval Research Laboratory
Washington 25, D.C.
ATTN: Mr. Robert E. Roberson

National Bureau of Standards
Computation Laboratories
Washington 25, D.C.
ATTN: Dr. John Todd

Dr. A.C. McNish
National Bureau of Standards
Washington 25, D.C.

Dr. Derrick H. Lehmer
Department of Mathematics
University of California
Berkeley, California

Professor Bernard Epstein
Department of Mathematics
Stanford University
Stanford, California

Division of Electrical Engineering
Electronics Research Laboratory
University of California
Berkeley 4, California
ATTN: Dr. Samuel Silver

Dr. Joseph Kaplan
Department of Physics
University of California
Los Angeles, California

Dr. A. Erdélyi
California Inst. of Technology
1201 E. California St.
Pasadena, California

Dr. Robert Kalaba
Electronics Division
Rand Corporation
Santa Monica, California

Stanford Research Institute
Stanford, California
ATTN: Dr. J.V.N. Granger
Head, Radio Systems Laboratory

Dr. Vic Twersky
Electronics Defense Laboratory
Box 205
Mountain View, California

Hughes Aircraft Company
Research and Development Library
Culver City, California
ATTN: M. Podner

Georgia Institute of Technology
Engineering Experimental Station
Atlanta, Georgia
ATTN: Dr. James E. Boyd

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Ave.
Silver Spring, Maryland
ATTN: Mr. F.T. McClure

Dr. Donald E. Kerr
Department of Physics
The Johns Hopkins University
Baltimore 18, Maryland

Dr. A. Weinstein
Institute of Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland

Professor J.A. Pierce
Harvard University
Cambridge 38, Mass.

Dr. F.M. Wiener
Bolt Beranek and Newman Inc.
16 Eliot St.
Cambridge 38, Mass.

Mrs. Marjorie L. Cox, Librarian
Technical Reports Collection
Room 303A, Pierce Hall
Harvard University
Cambridge 38, Mass.

Dr. Arthur A. Oliner
Microwave Research Institute
Polytechnic Institute of Brooklyn
55 Johnson St.
Brooklyn, N.Y.

Professor J.H. Mulligan
School of Engineering
New York University
New York, N.Y.

Professor Albert Heins
Carnegie Institute of Technology
Pittsburgh, Penna.

Professor Fred A. Ficken
University of Tennessee
Knoxville, Tennessee

Miss Barbara C. Grimes, Librarian
Federal Communications Commission
Washington 25, D.C.

Carnegie Institute of Washington
Department of Terrestrial Magnetism
5241 Broad Branch Road, N.W.
Washington 15, D.C.
ATTN: Library

Mathematical Reviews
190 Hope Street
Providence, Rhode Island

Mr. Martin Katzin
154 Fleetwood Terrace
Silver Spring, Maryland

Prof. B.H. Bissinger
Lebanon Valley College
Annville, Pennsylvania

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Dr. B.S. Gourary

Dr. Charles H. Papas
Dept. of Electrical Engineering
California Institute of Technology
Pasadena, California

Dr. Rodman Doll
311 W. Cross St.
Ypsilanti, Michigan

California Institute of Technology
Pasadena 4, California
ATTN: Mr. Calvin Wilcox

Technical Director
Combat Development Dept.
Army Electronic Proving Ground
Fort Huachuca, Arizona

National Bureau of Standards
Boulder, Colorado
ATTN: Dr. W.R. Gallet

Mrs. A.M. Gray
Engineering Library, Plant 5
Grumman Aircraft Corp.
Bethpage, L.I., N.Y.

Professor Bernard Friedman
Dept. of Mathematics
University of California
Berkeley, California

Dr. Jane Scanlon
224-03 67th Avenue
Bayside 64, N.Y.

Dr. Solomon L. Schwebel
3689 Louis Road
Palo Alto, California

Engineering Library
University of California
405 Hilgard Avenue
Los Angeles 24, California

University of Minnesota
The University Library
Minneapolis 14, Minnesota
ATTN: Exchange Division

Lincoln Laboratory
Massachusetts Institute of Technology
P.O. Box 73
Lexington 73, Massachusetts
ATTN: Dr. Shou Chin Wang, Rm. C-351

Technical Research Group
17 Union Square West
New York 3, N.Y.
ATTN: Dr. L. Goldmuntz

Hoffman Laboratories, Inc.
Advanced Development Section
3761 South Hill Street
Los Angeles 7, California
ATTN: Dr. Richard B. Farrar

Institute of Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland
ATTN: Dr. Elliott Montroll

Brandeis University
Waltham, Massachusetts
ATTN: Library

General Electric Company
Microwave Laboratory
Electronics Division
Stanford Industrial Park
Palo Alto, California
ATTN: Library

Dr. John B. Smyth
Smyth Research Associates
3930 4th Avenue
San Diego 3, California

Dr. Georges G. Weill
Electrical Engineering
California Institute of Technology
Pasadena, California

Naval Research Laboratory
Washington 25, D.C.
ATTN: Dr. Henry J. Passerini
Code 5278 A

Dr. George Kear
5 Culver Court
Orinda, California

California Institute of Technology
Electrical Engineering
Pasadena, California
ATTN: Dr. Zohrab A. Kaprielian

Prof. Nathan Marcuvitz
Microwave Research Institute
Brooklyn Polytechnic
55 Johnson Street
Brooklyn, N.Y.

Dr. Lester Kraus
4935 Whitehaven Way
San Diego 10, California

Dr. Jerry Shmoys
Dept. of Electrical Engineering
Brooklyn Polytechnic
85 Livingston Street
Brooklyn, N.Y.

Dr. I. Kolodner
Dept. of Mathematics
University of New Mexico
Albuquerque, New Mexico

Dr. Harry Hochstadt
Dept. of Mathematics
Polytechnic Institute of Brooklyn
Johnson and Jay Streets
Brooklyn, N.Y.

Hycon Eastern Inc.
75 Cambridge Parkway
Cambridge, Massachusetts
Attn: Seymour Stein

Mr. K.S. Kelleher, Section Head
Melpar, Inc.
3000 Arlington Boulevard
Falls Church, Virginia

Department of Engineering
Brown University
Providence, R.I.
ATTN: Dr. V.M. Papadopoulos

Wayne State University
Kresge-Hooker Science Library
5250 Second Boulvard
Detroit 2, Michigan

Dr. J. Brandstatten
Stanford Research Institute
S. Pasadena, California

ARRA
1 Bond Street
Westbury, L.I., N.Y.
ATTN: Dr. Norman Spector

Varian Associates
611 Hansen Way
Palo Alto, California
ATTN: (Mrs.) Perry Conway
Technical Librarian

Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Massachusetts
ATTN: Dr. Ming S. Wong, CRRKP

Professor C. A. Woonton, Director
Eaton Electronics Research Laboratory
McGill University
Montreal, Canada

Dr. D.S. Jones
University of North Staffordshire
Keele, Staffordshire
England

Science Abstracts
Institute of Electrical Engineers
Savoy Place
London W.C. 2, England
ATTN: R.M. Crowther

Dr. Jean-Claude Simon
Centre de Recherches techniques
Compagnie generale de T.S.F.
Paris 19, France

Professor A. van Wijngaarden
Mathematisch Centrum
2^e Boerhaavestraat 49
Amsterdam-Zuid, Holland

Dr. C.J. Bouwkamp
Philips Research Laboratories
Eindhoven, Netherlands

Dr. E.T. Copson
Department of Mathematics
United College
University of St. Andrews
St. Andrews, Scotland

Université de Paris
Cabinet du Department
des Sciences Mathématiques
Institut Henri Poincaré
11 Rue Pierre Curie
Paris 5^e, France

Dr. W. Elwyn Williams
Dept. of Applied Math.
University of Liverpool
Liverpool, England

University of Cambridge
Cavendish Laboratory
Cambridge, England
ATTN: Prof. Philip Clemmow

Technische Hogeschool
Instituut voor Toegepaste Wiskunde
Jaffalaan 162
Delft, Holland
ATTN: Prof. R. Timman

Instituto de Fisica Teorica
Rue Pamplona, 145
S. Paulo, Brasil

Dr. H. Bremmer
Philips Research Laboratories
Eindhoven, Netherlands

Electrotechnical Laboratory
Nagata-cho, Chiyoda-ku
Tokyo, Japan
ATTN: Library

Dr. A.T. De Hoop
Laboratorium voor Electrotechnik
Technische Hogeschool
Kanaalweg 2 B

Date Due

MAR 17 '60			
OCT 24 60			
[illegible]			
[illegible]			
[illegible]			
JUN 13 1971			
APR - 2 1979			
GB	PRINTED	IN U. S. A.	

NYU
BR-
30 Leis.
c. 1 On the Dirichlet problem
for the reduced wave equation.

NYU
BR-
30 Leis.
AUTHOR
c. 1
On the Dirichlet
TITLE problem for the reduced
wave equation.

DATE DUE	BORROWER'S NAME	ROOM NUMBER
MAR 17 '60	S Forrester	
[illegible] 60	~~Figueroa~~	
[illegible]	~~Ávila~~	

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.